# Technical Comments

## Comments on "A Note on the General Solution of the Two-Dimensional Linear Elasticity Problem in Polar Coordinates"

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IN a recent technical note, W. Z. Sadeh[1] claimed to have found four new solutions to the biharmonic equation. However, these solutions are not new. For example, the first three of them can be found in Ref. 2, and the fourth can easily be obtained from the third by a simple 90° rotation of the coordinate axes. Also, all four were given in Ref. 3.

There appear to be three kinds of solutions of the biharmonic compatibility equation: 1) those having periodic stress components and periodic displacements, 2) those having periodic stress components and nonperiodic displacements (called Volterra-type dislocations[4] if the origin lies within the body), and 3) those having nonperiodic stress components and nonperiodic displacements.

All three types were included in the stress functions in Refs. 2 and 3, whereas Ref. 5 included only the first two types. A derivation of the complete form for two-dimensional stress functions having periodic stress components was presented in Ref. 6.

Nonperiodic stress components have been discussed in detail in Refs. 7–9. As mentioned in Ref. 7, the difference between the Volterra-dislocation type of solution (type 2) and type 3 is that the former is associated with purely rigid-body displacements whereas the latter is associated with elastic displacements. Reference 8 discussed type 3 solutions under the condition that the traction on a stationary surface of discontinuity be continuous. It also pointed out that in the type 3 solution, the location and shape of the dislocational barrier must be specified. Reference 9 discussed type 3 solutions in which even the surface tractions were not required to be continuous.

It is interesting to note that Ref. 2 presented some solutions of the biharmonic equation which were not included in any of the other references. Recently, Ref. 10 has presented couple-stress-function solutions, which, in two-dimensional couple-stress elasticity, couple with all of the solutions of the biharmonic equation given in Ref. 2.

### References

[1] Sadeh, W. Z., "A note on the general solution of the two-dimensional linear elasticity problem in polar coordinates," AIAA J. 5, 354 (1967).

[2] Biezeno, C. B. and Grammel, R., *Engineering Dynamics, Vol. I-Theory of Elasticity; Analytical and Experimental Methods* (Blackie & Son Ltd., London, 1955), transl. of 2nd German ed. p. 158.

[3] Filonenko-Borodich, M., *Theory of Elasticity* (Foreign Languages Publishing House, Moscow, 1958), English transl., p. 216.

[4] Volterra, V., "Sur l'équilibre des corps élastiques multipliment connexes," Ann. Ecole Norm. Super. Ser. 3, 24, 401–517 (1907).

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[5] Timoshenko, S. and Goodier, J. N., *Theory of Elasticity* (McGraw-Hill Book Company Inc., New York, 1951), 2nd ed., p. 116.

[6] Bert, C. W., "Complete stress function for nonhomogeneous, anisotropic, plane problems in continuum mechanics," J. Aerospace Sci. 29, 756–757 (1962).

[7] Mann, E. H., "An elastic theory of dislocations," Proc. Roy. Soc. (London) A199, 376–394 (1949).

[8] Bogdanoff, J. L., "On the theory of dislocations," J. Appl. Phys. 21, 1258–1263 (1950).

[9] Goodier, J. N. and Wilhoit, J. C., Jr., "Elastic stress discontinuities in ring plates," *Proceedings of the 4th Midwestern Conference on Solid Mechanics, University of Texas* (Univ. of Texas, Austin, Texas, 1959), pp. 152–170.

[10] Bert, C. W. and Appl, F. J., "Two-dimensional couple-stress elasticity," *5th U. S. National Congress of Applied Mechanics, Minneapolis, Minn.* (American Society of Mechanical Engineers, New York, 1966).

## Comment on "A Note on the General Solution of the Two-Dimensional Linear Elasticity Problem in Polar Coordinates"

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THE author's[1] inference that he has obtained the general solution to the biharmonic equation is misleading. It is easy to show[2] that the general solution to $\nabla^4\Phi = 0$ is

$$\Phi = f_1(x + iy) + f_2(x - iy) + xf_3(x + iy) + xf_4(x - iy)$$

or, in polar coordinates,

$$\Phi = f_1(re^{i\theta}) + f_2(re^{-i\theta}) + r\cos\theta f_3(re^{i\theta}) + r\cos\theta f_4(re^{-i\theta})$$

where $f_1$, $f_2$, $f_3$, and $f_4$ are arbitrary functions of their arguments. It is apparent from the discussion by Timoshenko,[3] that the only terms he included in his so-called "general solution" are terms which represent solutions of known physical significance. In fact, Timoshenko includes in his expression the term $d_0r^2\theta$ which does not appear in Michell's[4] original work but which does represent the stress function for a known problem. Neither Timoshenko nor Michell indicate a mathematical procedure for arriving at their results.

A separation of variables solution to the biharmonic equation is presented in Ref. 5 where the solution is assumed to be of the form

$$\Phi = F(r)G(\theta)$$

with $G(\theta)$ represented by either $\sin n\theta$ or $\cos n\theta$. Introducing a new variable $t = \ln r$, and assuming that

$$F = Ke^{\lambda t}$$

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it is shown in Ref. 5 that solutions are obtained when

$$\lambda = n \qquad \lambda = -n \qquad \lambda = 2 + n \qquad \lambda = 2 - n \tag{1}$$

Because of the double roots which occur when $n = 0$ or $n = 1$, the solutions for $F(r)$ are shown to be of the following three forms:

$$\left.\begin{array}{ll} n = 0 & F(r) = a_0 + b_0 \ln r + c_0 r^2 + d_0 r^2 \ln r \\ n = 1 & F(r) = a_1 r + b_1/r + c_1 r^3 + d_1 r \ln r \\ n \geqslant 2 & F(r) = a_n r^n + b_n r^{-n} + c_n r^{2+n} + d_n r^{2-n} \end{array}\right\} \tag{2}$$

The stress functions obtained from these expressions do not include the following terms which appear in Timoshenko's "general solution":

$$\begin{array}{ll} \Phi_1 = d_0 r^2 \theta & \Phi_2 = a_0' \theta \\ \Phi_3 = (a_1/2) r\theta \sin\theta & \Phi_4 = -(c_1/2) r\theta \cos\theta \end{array} \tag{3}$$

In an unpublished M.S. thesis, Zuercher,[6] utilizing the same separation of variables approach as was later used in Ref. 5, showed that it is not sufficient to consider the multiple values of $\lambda$ in Eq. (1) which occur when $n$ takes on the values $n = 0$ and $n = 1$. To get a complete solution, it is also necessary to consider the multiple values of $n$ which occur when $\lambda$ takes on the values $\lambda = 0$ and $\lambda = 1$. This yields, in addition to all the terms in Eq. (2), the terms which lead to the stress function given in Eq. (3) plus the four "new" terms given by Sadeh.[1]

Zuercher obtained additional solutions to the biharmonic equation by assuming a separation of variables solution in the form

$$\Phi = F_1(r)G_1(\theta) + F_2(r)G_2(\theta)$$

where it was not necessary for each term itself to be biharmonic. In particular he considered

$$\Phi = r^n \ln r \begin{Bmatrix} \cos a\theta \\ \sin a\theta \end{Bmatrix} + r^n\theta \begin{Bmatrix} \sin a\theta \\ \cos a\theta \end{Bmatrix}$$

and this led to solutions of the form

$$\left.\begin{array}{l} \Phi_1 = f_n[r^n \ln r \cos n\theta - r^n\theta \sin n\theta] \\ \Phi_2 = g_n[r^n \ln r \cos(n-2)\theta - r^n\theta \sin(n-2)\theta] \\ \Phi_3 = h_n[r^n \ln r \sin n\theta + r^n\theta \cos n\theta] \\ \Phi_4 = j_n[r^n \ln r \sin(n-2)\theta + r^n\theta \cos(n-2)\theta] \end{array}\right\} \tag{4}$$

where, except for the special cases of $n = 0$ and $n = 1$, each expression in brackets must be taken in its entirety. With regard to the physical significance of any of these "new" terms, Hyman[7] showed that the stress functions for three problems listed in Timoshenko (problems 17, 20, and 21 of Chap. 4) when written in polar form contain terms that are included in Eq. (4).

### References


[1] Sadeh, W. Z., "A Note on the General Solution of the Two Dimensional Linear Elasticity Problem in Polar Coordinates," *AIAA Journal*, Vol. 5, No. 2, 1967, p. 354.

[2] Hildebrand, F. B., *Advanced Calculus for Applications*, Prentice-Hall, Englewood Cliffs, N.J., 1964, Chap. 8, p. 417.

[3] Timoshenko, S. and Goodier, J. N., *Theory of Elasticity*, 2nd ed., McGraw-Hill, New York, 1951, Chap. 4, p. 117.

[4] Michell, J. H., "Determination of Stress in an Elastic Solid with Application to the Theory of Plates," *Proceedings of the London Mathematical Society*, Vol. 31, 1899, p. 100.

[5] Durelli, A. J., Phillips, E. A., and Tsao, C. H., *Introduction to the Theoretical and Experimental Analysis of Stress and Strain*, McGraw-Hill, New York, 1958, p. 136.

[6] Zuercher, J. D., "A General Solution of the Two-Dimensional Stress Problem in Polar Coordinates," M.S. thesis, 1954, St. Louis University, St. Louis, Mo.

[7] Hyman, B. I., "On the General Form of the Airy Stress Function in Two Dimensions," M.S. thesis, 1961, St. Louis University, St. Louis, Mo.


## Comment on "A Note on the General Solution of the Two-Dimensional Linear Elasticity Problem in Polar Coordinates"


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THE additional solutions to the two-dimensional biharmonic equation in polar coordinates found by W. Z. Sadeh in Ref. 1 have been known for several decades! Applications of these solutions to slit plates and ring plate sectors can be found in Refs. 2, 3, and elsewhere.

### References


[1] Sadeh, W. Z., "A note on the general solution of the two-dimensional linear elasticity problem in polar coordinates," *AIAA Journal*, Vol. 5, No. 2, Feb. 1967, p. 354.

[2] Sonntag, R., "Über ein Problem der aufgeschnittenen Kreisringplatte," *Ingenieur Archiv*, Vol. 1, 1930, pp. 333–349.

[3] Mann, E. H., "An Elastic Theory of Dislocations," *Proceedings of the Royal Society (London), Series A*, Vol. 199, 1949, pp. 376–393.



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## Reply by Author to the Comments by C. W. Bert, B. I. Hyman, and F. Y. M. Wan


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THE author is indebted to C. W. Bert, B. I. Hyman, and Y. M. Wan for their useful and very pertinent comments. Unfortunately it was only after the publication of the Technical Note that the author became aware of an earlier similar solution by Filonenko-Borodich,[1] through a private communication from P. H. Francis (Senior Research Engineer, Department of Mechanical Sciences, Southwest Research Institute, San Antonio, Texas). In addition to this reference, the following bibliography is presented for those readers wishing to acquaint themselves further with the subject.

### References


[1] Filonenko-Borodich, M., *Theory of Elasticity*, Foreign Languages Publishing House, Moscow, 1958; English translation, Dover, 1965.

[2] Biezeno, C. B. and Grammel, R., *Engineering Dynamics, Vol. I—Theory of Elasticity; Analytical and Experimental Methods*, English translation of 2nd German ed., Blackie & Son, London, 1955.

[3] Goodier, J. N. and Wilhoit, J. C., Jr., "Elastic Stress Discontinuities in Ring Plates," *Proceeding of the Fourth Annual Conference on Solid Mechanics*, The University of Texas, Austin, Texas, Sept. 1959.

[4] Bert, C. W., "Complete Stress Function for Nonhomogeneous, Anisotropic, Plane Problems in Continuum Mechanics," *Journal of Aerospace Sciences*, Vol. 29, 1962, pp. 756–757.



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